A gentle introduction to the foundations of classical electrodynamics: The meaning of the excitations $(\mathcal{D}, \mathcal{H})$ and the field strengths (E, B)

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Abstract

The axiomatic structure of the electromagnetic theory is outlined. We will base classical electrodynamics on (1) electric charge conservation, (2) the Lorentz force, (3) magnetic flux conservation, and (4) on the Maxwell-Lorentz spacetime relations. This yields the Maxwell equations. The consequences will be drawn, inter alia, for the interpretation and the dimension of the electric and magnetic fields.

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I. INTRODUCTION

In Cologne, we teach a course on Theoretical Physics II (electrodynamics) to students of physics in their fourth semester. For several years, we have been using for that purpose the calculus of exterior differential forms, see [1,2], because we believe that this is the appropriate formalism: It is based on objects which possess a clear operational interpretation, it elucidates the fundamental structure of Maxwell's equations and their mutual interrelationship, and it invites a 4-dimensional representation appropriate for special and general relativity theory (i.e., including gravity, see [3,4]).

Our experimental colleagues are somewhat skeptical; and not only them. Therefore we were invited to give, within 90 minutes, a sort of popular survey of electrodynamics in exterior calculus to the members of one of our experimental institutes (group of H. Micklitz). The present article is a worked-out version of this talk. We believe that it could also be useful for other universities.

Subsequent to the talk we had given, we found the highly interesting and historically oriented article of Roche [5] on "B and H, the intensity vectors of magnetism...". Therein, the corresponding work of Bamberg and Sternberg [6], Bopp [7], Ingarden and Jamiołkowski [8], Kovetz [9], Post [10], Sommerfeld [11], and Truesdell and Toupin [12], to drop just a few names, was neglected yielding a picture of H and B which looks to us as being not up of date; one should also compare in this context the letter of Chambers [13] and the book of Roche [14], in particular its Chapter 9. Below we will suggest answers to some of Roche's questions.

Moreover, "...any system that gives E and B different units, when they are related through a relativistic transformation, is on the far side of sanity" is an apodictic statement of Fitch [15]. In the sequel, we will prove that we are on the far side of sanity: The absolute dimension of E turns out to be magnetic flux/time and that of B magnetic flux, see Sec. IV.

According to the audience we want to address, we will skip all mathematical details and take recourse to plausibility considerations. In order to make the paper self-contained, we

present though a brief summary of exterior calculus in the Appendix. A good reference to the mathematics underlying our presentation is the book of Frankel [16], see also [6] and [17]. For the experimental side of our subject we refer to Bergmann-Schaefer [18].

As a preview, let us collect essential information about the electromagnetic field in Table I. The explanations will follow below.

Table I. The electromagnetic field

Field	name	math.	independent	related	reflec-	absolute
		object	components	to	tion	dimension
\mathcal{D}	electric	odd	$\mathcal{D}_{23},\mathcal{D}_{31},\mathcal{D}_{12}$	area	$-\mathcal{D}$	q = electric
	excitation	2-form				charge
\mathcal{H}	magnetic	odd	$\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3$	line	$-\mathcal{H}$	q/t
	excitation	1-form				
E	electric	even	E_1, E_2, E_3	line	E	Φ_0/t
	field strength	1-form				
В	magnetic	even	B_{23}, B_{31}, B_{12}	area	В	$\Phi_0 = \text{mag-}$
	field strength	2-form				netic flux

It was Maxwell himself who advised us to be very careful in assigning a certain physical quantity to a mathematical object. As it turns out, the mathematical images of $\mathcal{D}, \mathcal{H}, E, B$ are all different from each other. This is well encoded in Schouten's images of the electromagnetic field in Fig.1.

FIGURES

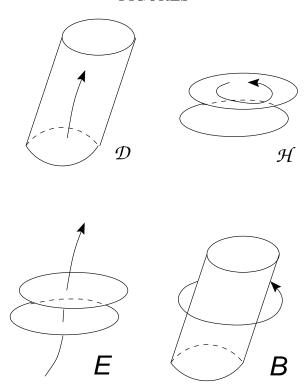


FIG. 1. Schouten's images of the electromagnetic field, see [17] p. 133.

II. ELECTRIC CHARGE CONSERVATION

The conservation of electric charge was already recognized as fundamental law during the time of Franklin (around 1750) well before Coulomb discovered his force law in 1785. Nowadays, when one can catch single electrons and single protons and their antiparticles in traps and can *count* them individually (see, e.g., Dehmelt [19], Paul [20], Devoret et al. [21], and Lafarge et al. [22]), we are more sure than ever that electric charge conservation is a valid fundamental law of nature. Therefore matter carries as a *primary quality* something called electric charge which only occurs in positive or negative units of an elementary charge e (or, in the case of quarks, of 1/3th of it) and which can be counted. Thus it is justified to introduce the physical dimension of charge q as a new and independent concept. Ideally one should measure a charge in units of e/3. However, for practical reasons, the SI-unit C (coulomb) is used in laboratory physics.

Let us start with the 3-dimensional Euclidean space in which we mark a 3-dimensional domain V. Hereafter, the local coordinates in this space will be denoted by x^a and the time by t, with the basis vectors $e_a := \partial_a$ and $a, b, \ldots = 1, 2, 3$, see Fig.2. The total charge in the domain V is given by the integral

$$Q = \int_{V} \rho \,, \tag{2.1}$$

where the electric charge density ρ is the 3-form $\rho = \frac{1}{3!} \rho_{abc} dx^a \wedge dx^b \wedge dx^c = \rho_{123} dx^1 \wedge dx^2 \wedge dx^3$. Here summation is understood over the indices a, b, c and $\rho_{abc} = -\rho_{bac} = \rho_{bca} = \dots$, i.e., the components ρ_{abc} of the charge density 3-form ρ are antisymmetric under the exchange of two indices, leaving only one independent component ρ_{123} . The wedge \wedge denotes the (anticommutative) exterior product of two forms, and $dx^1 \wedge dx^2 \wedge dx^3$ represents the volume "element". For our present purpose it is enough to know, for more details see the Appendix, that a 3-form (a p-form) is an object that, if integrated over a 3-dimensional (p-dimensional) domain, yields a scalar quantity, here the charge Q.

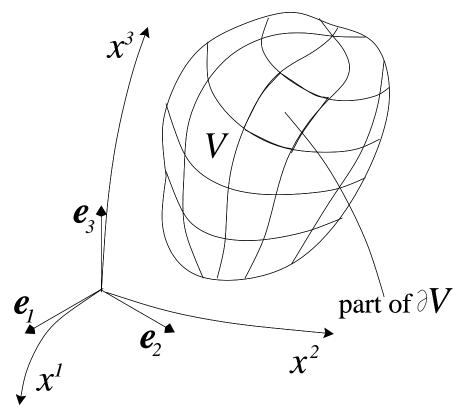


FIG. 2. A volume V with its boundary ∂V .

The dimension of Q is [Q] = q. Since an integral (a summation after all) cannot change the dimension, the dimension of the charge density 3-form and its components are, respectively, $[\rho] = q$, and $[\rho_{abc}] = q/\ell^3$, with $\ell = \text{length}$.

In general, the elementary charges are not at rest. The electric current J flowing across a 2-dimensional surface S is given by the integral

$$J = \int_{S} j, \qquad (2.2)$$

see Fig.3. Accordingly, the electric current density j turns out to be a 2-form: $j = \frac{1}{2!}j_{ab} dx^a \wedge dx^b = j_{12} dx^1 \wedge dx^2 + j_{13} dx^1 \wedge dx^3 + j_{23} dx^2 \wedge dx^3$, with $j_{ab} = -j_{ba}$. If t = time, then the dimensions of the current integral and the current 2-form and its components are [J] = [j] = q/t and $[j_{ab}] = q/(t \ell^2)$, respectively.

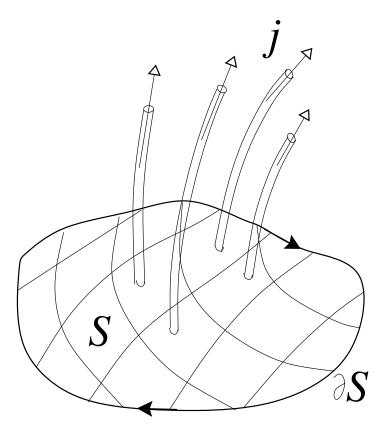


FIG. 3. A surface S with its boundary ∂S .

If we use the abbreviation $\partial_t := \partial/\partial t$, the global electric charge conservation can be expressed as

$$\partial_t \int_V \rho + \int_{\partial V} j = 0$$
 (Axiom 1), (2.3)

where the surface integral is evaluated over the (closed and 2-dimensional) boundary of V, which we denote by ∂V , see Fig.2. The change per time interval of the charge contained in V is balanced by the flow of the electric current through the boundary of the domain.

The closed surface integral $\int_{\partial V} j$ can be transformed into a volume integral $\int_{V} dj$ by Stokes's theorem (9.4). Here d denotes the exterior derivative that increases the rank of a form by one, i.e. dj is a 3-form. Thus (2.3) translates into

$$\int_{V} (\partial_t \rho + dj) = 0. \tag{2.4}$$

Since this is valid for an arbitrary domain, we arrive at the *local form* of electric charge conservation,

$$dj + \partial_t \rho = 0. (2.5)$$

III. EXCITATIONS

Since the charge density ρ is a 3-form, its exterior derivative vanishes: $d \rho = 0$. Then, by a theorem of de Rham, it follows that ρ can be derived from a "potential":

$$d\rho = 0 \implies \rho = d\mathcal{D}.$$
 (3.1)

In this way one finds the electric excitation 2-form \mathcal{D} . Its absolute dimension is $[\mathcal{D}] = [\rho] = q$, furthermore, for the components, $[\mathcal{D}_{ab}] = [\mathcal{D}]/\ell^2 = q/\ell^2$.

Substituting $(3.1)_2$ into charge conservation (2.5) and once again using the de Rham theorem, we find another "potential" for the current density:

$$d(j + \partial_t \mathcal{D}) = 0 \qquad \Longrightarrow \qquad j + \partial_t \mathcal{D} = d\mathcal{H}. \tag{3.2}$$

The magnetic excitation \mathcal{H} turns out to be a 1-form, see Fig.4. Its dimension is $[\mathcal{H}] = [j] = q/t$, $[\mathcal{H}_a] = q/(t\,\ell)$.

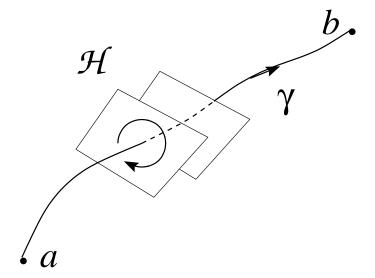


FIG. 4. A line γ with its boundary $\partial \gamma$, i.e., its end points a and b.

Consequently, the excitations $(\mathcal{D}, \mathcal{H})$ are potentials of the sources (ρ, j) . All these (additive) quantities (How much?) are described by odd differential forms.

In this way, we find the *inhomogeneous Maxwell* equations (the Gauss law and the Oersted-Ampère law):

$$d\mathcal{D} = \rho, \tag{3.3}$$

$$d\mathcal{H} - \partial_t \mathcal{D} = j. \tag{3.4}$$

Electric charge conservation is valid in microphysics. Therefore the corresponding Maxwell equations (3.3) and (3.4) are valid on the same "microphysical" level as the notions of charge density ρ and current density j. And with them the excitations \mathcal{D} and \mathcal{H} are microphysical quantities of the same type likewise – in contrast to what is stated in most textbooks.

Before we ever talked about *forces* on charges, charge conservation alone gave us the inhomogeneous Maxwell equations including the appropriate dimensions for the excitations \mathcal{D} and \mathcal{H} .

Under the assumption that \mathcal{D} vanishes inside an ideal electric conductor, one can get rid of the indeterminacy of \mathcal{D} which is inherent in the definition of the excitation as a "potential" of the charge density, and we can measure \mathcal{D} by means of two identical conducting plates

("Maxwellian double plates") which touch each other and which are *separated* in the \mathcal{D} field to be measured. The charge on one plate is then measured. Analogously, \mathcal{H} can be
measured by the Gauss compensation method or by a superconductor and the Meissner effect $(B=0 \to \mathcal{H}=0)$. Accordingly, the excitations do have a direct operational significance.

IV. FIELD STRENGTHS

So far, conserved charge was the notion at center stage. Now energy enters the scene, which opens the way for introducing the electromagnetic field strengths. Whereas the excitations $(\mathcal{D}, \mathcal{H})$ are linked to (and measured by) the charge and the current (ρ, j) , the electric and magnetic field strengths are usually introduced as forces acting on unit charges at rest or in motion, respectively.

Let us consider a point particle with electric charge e and velocity vector v. The force F acting on it is a 1-form since its (1-dimensional) line integral yields a scalar, namely the energy. Thus F carries the absolute dimension of an energy or of action over time: [F] = energy = h/t, where h denotes the dimension of an action. Accordingly, the local components F_a of the force $F = F_a dx^a$ possess the dimension $[F_a] = h/(t \ell) = \text{force}$.

In an electromagnetic field, the motion of a test particle is governed by the Lorentz force:

$$F = e(E - v \rfloor B) \qquad \text{(Axiom 2)}. \tag{4.1}$$

The symbol \rfloor denotes the interior product of a vector (here the velocity vector) with a p-form. It decreases the rank of a form by 1 (see the Appendix), and since $v\rfloor B$ is to be a 1-form, then B is a 2-form. Newly introduced by (4.1) are the electric field strength 1-form E and the magnetic field strength 2-form E. They are both *intensities* (How strong?).

The dimension of the velocity is [v] = 1/t. With the decomposition $v = v^a \partial_a$, we find for its components $[v^a] = \ell/t$. Then it is straightforward to read off from (4.1) the absolute dimension of the electric field strength $[E] = h/(tq) = \phi_0/t$, with $\phi_0 := h/q$. For its components we have $[E_a] = \phi_0/(t\ell)$. Analogously, for the dimension of the magnetic field

strength we find $[B] = (h/t)/(q/t) = h/q = \phi_0$ and $[B_{ab}] = \phi_0/\ell^2$, respectively. The field B carries the dimension of a magnetic flux ϕ_0 . In superconductors, magnetic flux can come in quantized flux tubes, so-called *fluxoids*, underlining the importance of the notion of magnetic flux.

The definition (4.1) of the field strengths makes sense only if the charge of the test particle is conserved. In other words, axiom 2 presupposes axiom 1 and should not be seen as a stand alone pillar of electrodynamics.

V. MAGNETIC FLUX CONSERVATION

Taking into account the rank (as exterior forms) of the field strengths, the only integral we can build up from E and B, respectively, are line integrals and surface integrals $\int_{\text{line}} E$ and $\int_{\text{surface}} B$. Apart from a factor t, the dimensions are equal. Hence, from a dimensional point of view, it seems sensible to postulate the conservation theorem (see Fig.5),

$$\partial_t \int_S B + \int_{\partial S} E = 0 \qquad \text{(Axiom 3)}.$$
 (5.1)

Magnetic flux conservation (5.1) has to be seen as an analog of electric charge conservation (2.3). Magnetic flux, though, is related to a 2-dimensional surface whereas electric charge is related to a 3-dimensional volume. Thus the integration domains in the conservation theorems (5.1) and (2.3) differ by one spatial dimension always.

Axiom 3 gains immediate evidence from the dynamics of an Abrikosov flux line lattice in a superconductor. There the quantized flux lines can be counted, they don't vanish nor are they created from nothing, rather they move in and out crossing the boundary ∂S of the surface S under consideration.

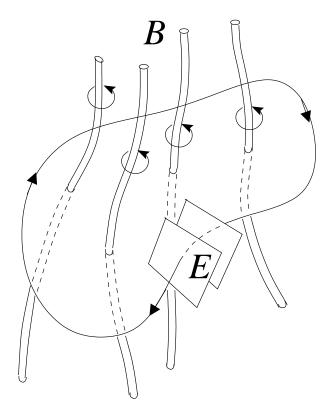


FIG. 5. Faraday's induction law.

Again, by means of Stokes's theorem (9.4), we can transform the boundary integral: $\int_{\partial S} E = \int_{S} dE$. Taking into account the arbitrariness of the surface S, we recover Faraday's induction law

$$dE + \partial_t B = 0, (5.2)$$

which is experimentally very well established. We differentiate Faraday's law by means of d and find $\partial_t(dB) = 0$. Since an integration constant other than zero is senseless (recall the relativity principle), we have

$$dB = 0. (5.3)$$

The homogeneous Maxwell equations (5.2) and (5.3) (Faraday's induction law and the closure of the magnetic field strength) nearly complete the construction of the theory.

We find the 3+3 time evolution equations (3.4) and (5.2) for the electromagnetic field $(\mathcal{D}, \mathcal{H}; E, B)$, i.e., for 6+6 components. Before we can find solutions of these equations,

we have to reduce the number of variables to 6, i.e., we have to cut them in half. Such a reduction is achieved by Axiom 4.

VI. THE MAXWELL-LORENTZ SPACETIME RELATIONS

In Axiom 4 we assume linear, isotropic, and centrosymmetric relations between the (additive) quantities and the intensities [23]:

$$\mathcal{D} = \varepsilon_0^* E$$
 and $\mathcal{H} = \frac{1}{\mu_0}^* B$ (Axiom 4). (6.1)

The proportionality coefficients ε_0 , μ_0 encode all the essential information about the electric and magnetic properties of spacetime. The Hodge star operator \star is needed, since we have to map a 1-form into a 2-form and vice versa or, more generally, a k-form into a (3-k)-form. Then the operator \star in (6.1) has the dimension of a length or its reciprocal. Note that the Hodge star depends on the metric of our Euclidean space, see the Appendix. Recalling the dimensions of the excitations and the field strengths, we find the dimensions of the electric constant ε_0 and the magnetic constant μ_0 as

$$[\varepsilon_0] = \frac{qt}{\phi_0 \ell} = \frac{1}{c \Omega_0}$$
 and $[\mu_0] = \frac{\phi_0 t}{q\ell} = \frac{\Omega_0}{c}$, (6.2)

respectively. They are also called vacuum permittivity and vacuum permeability, see the new CODATA report [24]. Here we define $\Omega_0 := \Phi_0/q = h/q^2$ and velocity $c := \ell/t$. Dimensionwise, it is clearly visible that

$$\left[\frac{1}{\sqrt{\varepsilon_0 \mu_0}}\right] = c \quad \text{and} \quad \left[\sqrt{\frac{\mu_0}{\varepsilon_0}}\right] = \Omega_0. \tag{6.3}$$

Obviously, the velocity c and the resistance Ω_0 are constants of nature, the velocity of light c being a universal one, whereas Ω_0 , the characteristic impedance (or wave resistance) of the vacuum [25], seemingly refers only to electromagnetic properties of spacetime. Note that $1/\Omega_0$ plays the role of the coupling constant of the electromagnetic field which enters as a factor into the free field Maxwell Lagrangian.

The Maxwell equations (3.3)-(3.4) and (5.2)-(5.3), together with the Maxwell-Lorentz spacetime (or aether) relations (6.1), constitute the foundations of classical electrodynamics. These laws, in the classical domain, are assumed to be of universal validity. Only if vacuum polarization effects of quantum electrodynamics are taken care of or hypothetical nonlocal terms should emerge from huge accelerations, Axiom 4 can pick up corrections yielding a nonlinear law (Heisenberg-Euler electrodynamics, see [4]) or a nonlocal law (Volterra-Mashhoon electrodynamics, see [26]), respectively. In this sense, the Maxwell equations are "more universal" than the Maxwell-Lorentz spacetime relations. The latter ones are not completely untouchable. We may consider (6.1) as constitutive relations for spacetime itself.

VII. SI-UNITS

The fundamental dimensions in the SI-system for mechanics and electrodynamics are $(\ell, t, M, q/t)$, with M as mass. And for each of those a unit was defined. However, since action – we denote its dimension by h – is a relativistic invariant quantity and since the electric charge is more fundamental than the electric current, we rather choose as the basic units

$$(\ell, t, h, q), \tag{7.1}$$

see Schouten [17] and Post [10]. Thus, instead of the kilogram and the ampere, we choose joule×second (or weber×coulomb) and the coulomb:

$$(m, s, Wb \times C, C). \tag{7.2}$$

Numerically, in the SI-system, one puts (for historical reasons)

$$\mu_0 = 4\pi \times 10^{-7} \frac{Wb \, s}{C \, m}$$
 (magnetic constant). (7.3)

Then measurements à la Weber-Kohlrausch vield

$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{C \, s}{Wb \, m}$$
 (electric constant). (7.4)

The SI-units of the electromagnetic field are collected in Table II.

Table II. SI-units of the electromagnetic field

Field	SI-unit of field	SI-unit of components of field
\mathcal{D}	C	C/m^2
\mathcal{H}	A = C/s	$A/m = C/(sm) \ (\rightarrow \text{oersted})$
E	Wb/s = V	V/m = Wb/(sm)
В	Wb	$Wb/m^2 = \text{tesla} = T \ (\rightarrow \text{gauss})$

VIII. ELECTRODYNAMICS IN MATTER

"It should be needless to remark that while from the mathematical standpoint a constitutive equation is a postulate or a definition, the first guide is physical experience, perhaps fortified by experimental data."

C. Truesdell and R.A. Toupin (1960)

In a great number of the texts on electrodynamics the electric and magnetic properties of media are described following the *macroscopic averaging* scheme of Lorentz (1916). However, this formalism has a number of serious limitations, see the relevant criticism of Hirst [27], e.g.. An appropriate modern presentation of this subject has been given in the textbook of Kovetz [9].

Here we follow a consistent *microscopic* approach to the electrodynamics in media, cf. [27]. The total charge or current density is the sum of the two contributions originating "from the inside" and "from the outside" of the medium:

$$\rho = \rho^{\text{mat}} + \rho^{\text{ext}}, \qquad j = j^{\text{mat}} + j^{\text{ext}}. \tag{8.1}$$

Hereafter, the bound charge [28] in matter is denoted by mat and the external charge [29] by ext. The same notational scheme will also be applied to the excitations \mathcal{D} and \mathcal{H} . Bound charge and bound current are inherent characteristics of matter determined by the medium itself. They vanish outside the medium. In contrast, external charge and external current in general do not vanish outside matter. They can be prepared for a specific purpose (such as

the scattering of a current of particles by a medium or a study of the reaction of a medium in response to a prescribed configuration of charges and currents).

We assume that the charge bound by matter fulfills the usual conservation law:

$$dj^{\text{mat}} + \partial_t \rho^{\text{mat}} = 0. (8.2)$$

Taking into account (2.5), this means that there is no physical exchange (or conversion) between the bound and the external charges. As a consequence of this assumption, we can repeat the arguments of Sec.III that will give rise to the corresponding excitations \mathcal{D}^{mat} and \mathcal{H}^{mat} as "potentials" for the bound charge and the bound current. The conventional names for these newly introduced excitations are polarization P and magnetization M, i.e.,

$$\mathcal{D}^{\text{mat}} \equiv -P, \qquad \mathcal{H}^{\text{mat}} \equiv M. \tag{8.3}$$

The minus sign is conventional, see Kovetz [9]. Thus, in analogy to the inhomogeneous Maxwell equations, we find

$$-dP = \rho^{\text{mat}}, \qquad dM + \partial_t P = j^{\text{mat}}. \tag{8.4}$$

The identifications (8.3) are only true up to an exact form. However, if we require $\mathcal{D}^{\text{mat}} = 0$ for E = 0 and $\mathcal{H}^{\text{mat}} = 0$ for B = 0, as we will do in (8.8), uniqueness is guaranteed.

The Maxwell equations are *linear* partial differential equations. Therefore we can define

$$\mathcal{D}^{\text{ext}} := \mathcal{D} - \mathcal{D}^{\text{mat}} = \mathcal{D} + P, \qquad \mathcal{H}^{\text{ext}} := \mathcal{H} - \mathcal{H}^{\text{mat}} = \mathcal{H} - M.$$
 (8.5)

The external excitations $(\mathcal{D}^{\text{ext}}, \mathcal{H}^{\text{ext}})$ can be understood as auxiliary quantities. In terms of these quantities, the *inhomogeneous* Maxwell equations for matter finally read:

$$d\mathcal{D}^{\text{ext}} = \rho^{\text{ext}}, \qquad \mathcal{D}^{\text{ext}} = \varepsilon_0^* E + P[E, B],$$
 (8.6)

$$d\mathcal{H}^{\text{ext}} - \partial_t \mathcal{D}^{\text{ext}} = j^{\text{ext}}, \qquad \mathcal{H}^{\text{ext}} = \frac{1}{\mu_0} *B - M[B, E].$$
 (8.7)

Here the polarization P[E, B] is a functional of the electromagnetic field strengths E and B. In general, it can depend also on the temperature T, and possibly of other thermodynamic variables specifying the material continuum under consideration; similar remarks apply to the magnetization M[B, E]. The system $(8.6)_1$ and $(8.7)_1$ looks similar to (3.3) and (3.4). However, the former equations refer only to the external fields and sources. The homogeneous Maxwell equations (5.2) and (5.3) remain valid in their original form.

In the simplest cases, we have the linear constitutive laws

$$P = \varepsilon_0 \chi_{\rm E} {}^{\star}E, \qquad M = \frac{1}{\mu_0} \chi_{\rm B} {}^{\star}B, \qquad (8.8)$$

with the electric and magnetic [30] susceptibilities (χ_E, χ_B). With material constants

$$\varepsilon := \varepsilon_0 \left(1 + \chi_{\mathcal{E}} \right), \qquad \mu := \frac{\mu_0}{1 - \chi_{\mathcal{B}}}, \tag{8.9}$$

one can rewrite the material laws (8.8) as

$$D^{\text{ext}} = \varepsilon^* E, \qquad H^{\text{ext}} = \frac{1}{\mu} {}^* B.$$
 (8.10)

For the discussion of the concrete applications of the developed microscopic theory in modern condensed matter physics, we refer to the review of Hirst [27].

IX. CONCLUSION

The Maxwell equations

$$d\mathcal{D} = \rho, \qquad d\mathcal{H} - \partial_t \mathcal{D} = j,$$
 (9.1)

$$dB = 0, dE + \partial_t B = 0, (9.2)$$

are the cornerstones of any classical theory of electromagnetism. As an expression of charge and flux conservation, they carry a high degree of plausibility as well as solid experimental support. The Maxwell equations in this form remain valid in an accelerated reference frame and in a gravitational field likewise, without any change.

The Maxwell-Lorentz spacetime relations

$$\mathcal{D} = \frac{1}{c\Omega_0} {}^*E \quad \text{and} \quad \mathcal{H} = \frac{c}{\Omega_0} {}^*B$$
 (9.3)

are necessary for developing the Maxwellian system into a predictive physical theory. They depend, via the star operator, on the metric of space and are, accordingly, influenced by the gravitational field. They are valid in very good approximation, but there are a few exceptions known (if the Casimir effect is to be described, e.g.).

For the description of matter, the sources (ρ, j) and the excitations $(\mathcal{D}, \mathcal{H})$ have to be split suitably in order to derive, from the microscopic equations (9.1) and (9.2), appropriate macroscopic expressions.

Summing up, we can give an answer to one of the central questions posed by Roche [5]: The need for the different notations and different dimensions and units for the excitation \mathcal{H} and the field strength B (and, similarly, for \mathcal{D} and E) is well substantiated by the *very different* geometrical properties and physical origins of these fields, see Table I and Fig. 1. Even in vacuum, these differences do *not* disappear.

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APPENDIX: THE ABC OF EXTERIOR CALCULUS

The formalism of exterior differential forms is widely used in different domains of mathematics and theoretical physics. In particular, in electromagnetic theory, exterior calculus offers an elegant and transparent framework for the introduction of the basic notions and for the formulation of the corresponding laws. Here, we will give a very elementary description of the objects and operations used above.

We will confine ourselves only to the case of a 3-dimensional space. Let be given the set of local coordinates $x^a = \{x^1, x^2, x^3\}$; hereafter Latin indices a, b, \ldots will run over 1, 2, 3. Then the vectors $e_a = \{\partial_1, \partial_2, \partial_3\}$ will serve as a basis of the tangent vector space at every point. The symbol dx^a denotes the set of linear 1-forms dual to the coordinate vector basis, $dx^a(e_b) = \delta_b^a$. An arbitrary k-form can be described, in local coordinates, by its components: $\varphi = \varphi_a dx^a = \varphi_1 dx^1 + \varphi_2 dx^2 + \varphi_3 dx^3$ for 1-forms and $\omega = \frac{1}{2} \omega_{ab} dx^a \wedge dx^b = \omega_{12} dx^1 \wedge dx^2 + \omega_{23} dx^2 \wedge dx^3 + \omega_{31} dx^3 \wedge dx^1$ for 2-forms. Any 3-form has a single nontrivial component, $\eta = \eta_{123} dx^1 \wedge dx^2 \wedge dx^3$. When η is smoothly defined on the whole space, it is called a volume form. Zero-forms are just the ordinary differentiable functions.

It is often stated that the exterior product " \wedge " generalizes the vector product. However, one should be careful with such statements, because the vector product in the standard 3-dimensional analysis is, strictly speaking, a superposition of the wedge product and of the Hodge duality operator. Thus, the vector product necessarily involves the metric on the manifold. In contrast, the exterior product is a pre-metric operation, although it resembles the vector product. For example, the exterior product of the two 1-forms ω and φ with the components ω_a and φ_a yields a 2-form $\omega \wedge \varphi$ with the local components $\{(\omega_2\varphi_3 - \omega_3\varphi_2), (\omega_3\varphi_1 - \omega_1\varphi_3), (\omega_1\varphi_2 - \omega_2\varphi_1)\}$.

The exterior differential d increases the rank of a form by 1. It is most easily described in local coordinates, see Table III. Thus, d naturally generalizes the "grad" operator which leads from a scalar to a vector and, at the same time, it represents a pre-metric extension of the "curl" operator. The exterior differential is a nilpotent operator, i.e., dd = 0.

Table III. Operators acting on an exterior form

	k-form $\omega = \frac{1}{k!} \omega_{a_1a_k} dx^{a_1} \wedge \ldots \wedge dx^{a_k}$, with $k = 0, 1, 2, 3$
d	(k+1)-form $d\omega = \frac{1}{(k+1)!} \left(\partial_{[a_1} \omega_{a_1 \dots a_{k+1}]} \right) dx^{a_1} \wedge \dots \wedge dx^{a_{k+1}}$
]	$(k-1)$ -form $v \rfloor \omega = \frac{1}{(k-1)!} v^a \omega_{aa_1a_{k-1}} dx^{a_1} \wedge \ldots \wedge dx^{a_{k-1}}$
*	$(3-k)\text{-form }^{\star}\omega = \frac{1}{k!}\omega_{a_1\dots a_k}g^{a_1b_1}\dots g^{a_kb_k}e_{b_k}\rfloor\dots\rfloor e_{b_1}\rfloor\eta$

Complementary to d, one can define an operation which decreases the rank of a form

by 1. This is the *interior product* of a vector with a k-form. Given the vector v with the components v^a , the interior product with the coframe 1-form yields $v\rfloor dx^a = v^a$, which is a sort of a projection along v. By linearity, the interior product of v with a k-form is defined as described in Table III.

The Hodge dual operator * maps k-forms into (3-k)-forms. Its introduction necessarily requires the metric which assigns a real number g(u,v)=g(v,u) to every two vectors u and v. In local coordinates, the components of the metric tensor are determined as the values of the scalar product of the basis vectors, $g_{ab}:=g(e_a,e_b)$. This matrix is positive definite. The metric introduces a natural volume 3-form $\eta:=\sqrt{\det g_{ab}}\,dx^1\wedge dx^2\wedge dx^3$ which underlies the definition of the Hodge operator *. The general expression is displayed in Table III. Explicitly the Hodge dual of the coframe 1-form reads, for example: ${}^*dx^a=\sqrt{\det g_{ab}}\,(g^{a1}\,dx^2\wedge dx^3+g^{a2}\,dx^3\wedge dx^1+g^{a3}\,dx^1\wedge dx^2)$, where g^{ab} is inverse to g_{ab} .

The notions of the *odd* and *even* exterior forms are closely related to the orientation of the manifold. In simple terms, these two types of forms are distinguished by their different behavior with respect to a reflection (i.e., a change of orientation): an even (odd) form does not change (changes) sign under a reflection transformation. These properties of odd and even forms are crucial in the integration theory, see, e.g., [16].

For a k-form an *integral* over a k-dimensional subspace is defined. For example, a 1-form can be integrated over a curve, a 2-form over a 2-surface, and a volume 3-form over the whole 3-dimensional space. We will not enter into the details here, limiting ourselves to the formulation of Stokes's theorem which occupies a central place in integration theory:

$$\int_{\partial C} \omega = \int_{C} d\omega. \tag{9.4}$$

Here ω is an arbitrary k-form, and C is an arbitrary (k+1)-dimensional (hyper)surface with the boundary ∂C .

For a deeper and a more rigorous introduction into exterior calculus, see, e.g., [6,16].

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- [28] Also called polarization charge.
- [29] Also called free, true, or real charge.
- [30] In older texts, the magnetization M was usually expressed in terms of H, namely $M = \chi_{\rm H} H$. For reasons of relativistic invariance, this is inappropriate, provided we start with $P = \varepsilon_0 \chi_{\rm E} *E$, as we do in (8.8)₂. Note that $\mu = \mu_0 (1 + \chi_{\rm H})$.